

# Contractions, deformations and curvature

Ángel Ballesteros<sup>a</sup>, Francisco J. Herranz<sup>a</sup>, Orlando Ragnisco<sup>b</sup> and Mariano Santander<sup>c</sup>

<sup>a</sup>*Departamento de Física, Universidad de Burgos, E-09001 Burgos, Spain*  
e-mail: angelb@ubu.es, fjherranz@ubu.es

<sup>b</sup>*Dipartimento di Fisica, Università di Roma Tre and Istituto Nazionale di Fisica Nucleare, Via Vasca Navale 84, I-00146 Rome, Italy*  
e-mail: ragnisco@fis.uniroma3.it

<sup>c</sup>*Departamento de Física Teórica, Universidad de Valladolid, E-47011 Valladolid, Spain*  
e-mail: msn@fta.uva.es

## Abstract

The role of curvature in relation with Lie algebra contractions of the pseudo-orthogonal algebras  $so(p, q)$  is fully described by considering some associated symmetrical homogeneous spaces of constant curvature within a Cayley–Klein framework. We show that a given Lie algebra contraction can be interpreted geometrically as the zero-curvature limit of some underlying homogeneous space with constant curvature. In particular, we study in detail the contraction process for the three classical Riemannian spaces (spherical, Euclidean, hyperbolic), three non-relativistic (Newtonian) spacetimes and three relativistic ((anti-)de Sitter and Minkowskian) spacetimes. Next, from a different perspective, we make use of quantum deformations of Lie algebras in order to construct a family of spaces of non-constant curvature that can be interpreted as deformations of the above nine spaces. In this framework, the quantum deformation parameter is identified as the parameter that controls the curvature of such “quantum” spaces.

**KEYWORDS:** Lie algebras, quantum groups, contraction, curvature, deformation, hyperbolic, de Sitter

**PACS:** 02.20.Sv 02.20.Uw 02.40.Ky

# 1 Introduction

Nowadays, contraction of Lie algebras is a well established theory that focuses the interest of both mathematicians and physicists. We recall that Lie algebra contractions began to be systematically formulated from the early works of Segal [1], Inönü and Wigner [2] and Saletan [3] (see also [4, 5, 6] and references therein). Roughly speaking, the way to obtain a contracted Lie algebra  $g'$  from an initial one  $g$  is to define the generators of  $g'$  in terms of those of  $g$  in an “adequate” form by introducing a contraction parameter  $\varepsilon$  in such a manner that under the limit  $\varepsilon \rightarrow 0$  the commutation relations of  $g$  reduce to those of  $g'$ .

Two well known examples of contraction, related with spaces with arbitrary dimension  $N$  are: (i) the *flat* contraction that goes from  $so(N+1)$  to the Euclidean algebra  $iso(N)$  and (ii) the *non-relativistic* contraction that transforms the Poincaré algebra  $iso(N-1, 1)$  into the Galilean one  $iiso(N-1)$ . When looking at the underlying symmetrical homogeneous spaces associated to the above Lie algebras, one finds that these contractions can geometrically be interpreted in terms of the vanishing of some (*constant*) *curvature* of such spaces [7]. The former example relates the  $N$ -dimensional (ND) spherical space  $SO(N+1)/SO(N)$  of constant curvature  $+1/R^2$  ( $R$  is the radius of the sphere) to the flat Euclidean one  $ISO(N)/SO(N)$ , that is, the limit  $\varepsilon \rightarrow 0$  corresponds to  $R \rightarrow \infty$ . The latter contraction when read as the ordinary non-relativistic limit, relates two *flat* spaces (Minkowskian versus Galilean spacetimes) but alternatively it can also be naturally interpreted as a contraction starting from the  $2(N-1)$ D space of (time-like) lines  $ISO(N-1, 1)/(\mathbb{R} \otimes SO(N-1))$  in the flat Minkowskian spacetime  $ISO(N-1, 1)/SO(N-1, 1)$ , to the flat space of worldlines  $IISO(N-1)/(\mathbb{R} \otimes SO(N-1))$  in the flat Galilean spacetime  $IISO(N-1)/ISO(N-1)$  under the limit  $c \rightarrow \infty$ . The natural curvature of the space of lines in Minkowskian spacetime turns out to be non-vanishing and equal to  $-1/c^2$ , where  $c$  is the relativistic constant, the speed of light. This interpretation of Lie algebra contractions in terms of zero-curvature limits for homogeneous spaces can be widely applied for many other cases, which fully cover the set of possible contractions within the four Cartan families of real semisimple Lie algebras.

On the other hand, let us consider a *quantum deformation* of the Lie algebra  $g$  endowed with a Hopf structure [8, 9, 10, 11], that is, a quantum algebra  $U_z(g)$  which is an algebra of formal power series in a deformation parameter  $z$  ( $q = e^z$ ) with coefficients in  $U(g)$ . In this case we know that if a Lie algebra contraction  $g \rightarrow g'$  exists under the limit  $\varepsilon \rightarrow 0$ , then this contraction limit can also be implemented at the deformed level  $U_z(g) \rightarrow U_{z'}(g')$  through a Lie bialgebra contraction [12]. The latter keeps the same contraction map for the generators while adds some transformation for the contracted deformation parameter  $z' = z/\varepsilon^n$ , where  $n$  is a real number to be fixed for each specific algebra and contraction. This process is rather similar to the so called generalized Inönü–Wigner contractions [4]. By following this approach, for the two aforementioned contractions one finds that  $U_z(so(N+1)) \rightarrow U_{z'}(iso(N))$  and  $U_z(iso(N-1, 1)) \rightarrow U_{z'}(iiso(N-1))$  under the limit  $\varepsilon \rightarrow 0$ .

We stress that a quantum deformation of the Lie algebra  $g$  provides an extra “quantity” in the underlying symmetry structure, the deformation parameter  $z$ , which can be interpreted in different ways depending on the specific model under consideration. For instance, as a fundamental scale in “generalizations” of the special relativity theory [13, 14], as a lattice step in relation with discretized symmetries [15], as a coupling constant in  $N$ -body problems [16], etc. However, one has to pay the price of losing the Lie structure

and therefore the corresponding geometric interpretation provided by the associated Lie group and their homogeneous spaces. Hence, in principle, the geometrical interpretation of “quantum” contractions in terms of curvatures seems to be lost.

Nevertheless, if the Lie algebra contraction procedure is read in the reverse direction as a Lie algebra deformation [17] or *classical deformation*, a new interpretation for quantum deformations arises in a “natural” way. It turns out that the classical deformation  $g' \rightarrow g$  (more precisely, from  $U(g')$  to  $g$ ) can be interpreted as the introduction of a constant curvature in a formerly flat homogeneous space associated to both  $g$  and  $g'$  [7, 18]. Such a deformation process can be iterated until when one arrives to a semisimple Lie algebra  $g$ , for which the associated homogeneous spaces (of points, lines, 2-planes, etc.) are endowed with a non-zero constant curvature. Consequently, since quantum algebras go beyond Lie algebras (generalizing them) the above ideas suggest that a quantum deformation might also be understood as the introduction of some kind of curvature in an appropriate context. In fact, we have recently shown [19, 20] that a quantum deformation does indeed introduce curvature on a certain flat space, but now this curvature is generically *non-constant* and is governed by the deformation parameter  $z$ . As a straightforward consequence, the non-deformed limit  $z \rightarrow 0$ , under which  $U_z(g) \rightarrow U(g) \sim g$ , can also be understood as a zero-curvature limit, *i.e.* as a contraction process.

The aim of this paper is to give a brief overview of a global and unified scheme for contractions and classical/quantum deformations in relation with curved spaces. Both subjects usually appear as *two* completely separate frameworks in the literature: contractions/classical deformations versus quantum deformations. For this purpose we shall choose a relevant family of *nine* spaces of constant curvature, namely: the Riemannian (spherical, Euclidean and hyperbolic), semi-Riemannian (non-relativistic oscillating/expanding Newton–Hooke (NH) and Galilean), and pseudo-Riemannian (relativistic (anti-)de Sitter and Minkowskian) spaces. Such a family can be described in a common setting by making use of the so called orthogonal Cayley–Klein (CK) or quasi-simple Lie algebras [21, 22, 23, 24, 25, 26, 27, 28], all of whose members are contractions of  $so(N+1)$ .

The scheme of the paper is as follows. In the next Section we recall the structure of the CK algebras, we also construct, by using a group theoretical approach, a family of symmetrical homogeneous spaces associated to each CK algebra, and next we explicitly obtain the above nine spaces in terms of geodesic polar (spherical) coordinates. In this way the role of curvature in the context of Lie algebra contractions/deformations is highlighted. In Section 3, we start from the non-standard quantum deformation of  $sl(2, \mathbb{R})$  [29],  $U_z(sl(2, \mathbb{R}))$ , written as a deformed Poisson algebra and, by taking three copies of  $U_z(sl(2, \mathbb{R}))$ , we are able to construct an infinite family of 3D deformed spaces endowed, in general, with a non-constant (scalar) curvature. A further change of coordinates allows us to introduce spherical coordinates in such a manner that we find that such deformed spaces are just non-constant curvature counterparts of the homogeneous spaces described in Section 2. In this way the relationships between curvature and contraction/quantum deformations can be explicitly analysed. Finally, the last Section contains some remarks and open problems.

## 2 Contraction, curvature and Lie algebras

### 2.1 Orthogonal CK algebras

Let us consider the real Lie algebra  $so(N+1)$  whose  $\frac{1}{2}N(N+1)$  generators  $J_{ab}$  ( $a, b = 0, 1, \dots, N$ ,  $a < b$ ) satisfy the non-vanishing Lie brackets given by

$$[J_{ab}, J_{ac}] = J_{bc}, \quad [J_{ab}, J_{bc}] = -J_{ac}, \quad [J_{ac}, J_{bc}] = J_{ab}, \quad a < b < c. \quad (2.1)$$

A fine grading group  $\mathbf{Z}_2^{\otimes N}$  of  $so(N+1)$  is spanned by the following  $N$  commuting involutive automorphisms  $\Theta^{(m)}$  ( $m = 1, \dots, N$ ) of (2.1):

$$\Theta^{(m)}(J_{ab}) = \begin{cases} J_{ab}, & \text{if either } m \leq a \text{ or } b < m; \\ -J_{ab}, & \text{if } a < m \leq b. \end{cases} \quad (2.2)$$

By applying the graded contraction theory [30, 31] a large family of contracted real Lie algebras can be obtained from  $so(N+1)$ ; this depends on  $2^N - 1$  real contraction parameters [32] which includes from the simple pseudo-orthogonal algebras  $so(p, q)$  (the  $B_l$  and  $D_l$  Cartan series) (when all the contraction parameters are different from zero) up to the Abelian algebra at the opposite case (when all the parameters are equal to zero). Certainly, properties associated with the simplicity of the algebra are lost at some point beyond the simple algebras in the contraction sequence. However there exists a particular subset of contracted Lie algebras which are “close to” to the simple ones [33], whose members are called CK or quasi-simple orthogonal algebras [24, 25, 26, 27]. For instance, all the CK algebras share, in any dimension, the same *rank* defined as the number of (functionally independent) Casimir invariants [28]. These are precisely called CK algebras since they are exactly the family of motion algebras of the geometries of a real space with a projective metric in the CK sense [21, 22].

This orthogonal CK family, here denoted  $so_\kappa(N+1)$ , depends on  $N$  real contraction coefficients  $\kappa = (\kappa_1, \dots, \kappa_N)$  and, as the essential trait is the sign of each  $\kappa_m$ , comprises  $3^N$  Lie algebras (up to isomorphisms the number is lesser, as CK algebras with different choices of signs for the set  $\kappa$  may turn out to be isomorphic). The non-zero commutators read [33]:

$$[J_{ab}, J_{ac}] = \kappa_{ab} J_{bc}, \quad [J_{ab}, J_{bc}] = -J_{ac}, \quad [J_{ac}, J_{bc}] = \kappa_{bc} J_{ab}, \quad a < b < c, \quad (2.3)$$

without sum over repeated indices and where the two-index coefficients  $\kappa_{ab}$  are expressed in terms of the  $N$  basic ones through

$$\kappa_{ab} = \kappa_{a+1} \kappa_{a+2} \cdots \kappa_b, \quad a, b = 0, 1, \dots, N, \quad a < b. \quad (2.4)$$

Each non-zero real coefficient  $\kappa_m$  can be reduced to either  $+1$  or  $-1$  by a rescaling of the Lie generators. The case  $\kappa_m = 0$  can be interpreted as an Inönü–Wigner contraction [2], with parameter  $\varepsilon_m \rightarrow 0$ , and defined by the map (cf. (2.2)):

$$\Gamma^{(m)}(J_{ab}) = \begin{cases} J_{ab}, & \text{if either } m \leq a \text{ or } b < m; \\ \varepsilon_m J_{ab}, & \text{if } a < m \leq b. \end{cases} \quad (2.5)$$

Each involution  $\Theta^{(m)}$  (2.2) provides a Cartan-like decomposition as a direct sum of anti-invariant and invariant subspaces, denoted  $p^{(m)}$  and  $h^{(m)}$ , respectively:

$$so_\kappa(N+1) = p^{(m)} \oplus h^{(m)}, \quad (2.6)$$

with the linear sum referring to the linear structure; Lie commutators fulfil:

$$[h^{(m)}, h^{(m)}] \subset h^{(m)}, \quad [h^{(m)}, p^{(m)}] \subset p^{(m)}, \quad [p^{(m)}, p^{(m)}] \subset h^{(m)}, \quad (2.7)$$

and thus  $h^{(m)}$  is always a Lie subalgebra with a direct sum structure:

$$h^{(m)} = so_{\kappa_1, \dots, \kappa_{m-1}}(m) \oplus so_{\kappa_{m+1}, \dots, \kappa_N}(N+1-m), \quad (2.8)$$

while the vector subspace  $p^{(m)}$  is generally not a subalgebra.

The decomposition (2.6) can be visualized in array form as follows:

$$\begin{array}{cccc|cccc} J_{01} & J_{02} & \dots & J_{0\,m-1} & J_{0m} & J_{0\,m+1} & \dots & J_{0N} \\ & J_{12} & \dots & J_{1\,m-1} & J_{1m} & J_{1\,m+1} & \dots & J_{1N} \\ & & \ddots & \vdots & \vdots & \vdots & & \vdots \\ & & & J_{m-2\,m-1} & J_{m-2\,m} & J_{m-2\,m+1} & \dots & J_{m-2\,N} \\ & & & & J_{m-1\,m} & J_{m-1\,m+1} & \dots & J_{m-1\,N} \\ & & & & & J_{m\,m+1} & \dots & J_{m\,N} \\ & & & & & & \ddots & \vdots \\ & & & & & & & J_{N-1\,N} \end{array} \quad (2.9)$$

The subspace  $p^{(m)}$  is spanned by the  $m(N+1-m)$  generators inside the rectangle; the left and down triangles correspond, in this order, to the subalgebras  $so_{\kappa_1, \dots, \kappa_{m-1}}(m)$  and  $so_{\kappa_{m+1}, \dots, \kappa_N}(N+1-m)$  of  $h^{(m)}$  (2.8).

As some relevant members contained within  $so_{\kappa}(N+1)$  we point out [28]:

- When all  $\kappa_a \neq 0 \, \forall a$ ,  $so_{\kappa}(N+1)$  is a (pseudo-)orthogonal algebra  $so(p, q)$  ( $p+q = N+1$ ) and  $(p, q)$  are the number of positive and negative terms in the invariant quadratic form with matrix  $(1, \kappa_{01}, \kappa_{02}, \dots, \kappa_{0N})$ .
- When  $\kappa_1 = 0$  we recover the inhomogeneous algebras with semidirect sum structure

$$so_{0, \kappa_2, \dots, \kappa_N}(N+1) \equiv t_N \odot so_{\kappa_2, \dots, \kappa_N}(N) \equiv iso(p, q), \quad p+q = N,$$

where the Abelian subalgebra  $t_N$  is spanned by  $\langle J_{0b}; b = 1, \dots, N \rangle$  and  $so_{\kappa_2, \dots, \kappa_N}(N)$  preserves the quadratic form with matrix  $\text{diag}(+, \kappa_{12}, \dots, \kappa_{1N})$ .

- When  $\kappa_1 = \kappa_2 = 0$  we get a “twice-inhomogeneous” pseudo-orthogonal algebra

$$so_{0,0, \kappa_3, \dots, \kappa_N}(N+1) \equiv t_N \odot (t_{N-1} \odot so_{\kappa_3, \dots, \kappa_N}(N-1)) \equiv iiso(p, q), \quad p+q = N-1,$$

where the metric of the subalgebra  $so_{\kappa_3, \dots, \kappa_N}(N-1)$  is  $(1, \kappa_{23}, \kappa_{24}, \dots, \kappa_{2N})$ .

- When  $\kappa_a = 0, a \notin \{1, N\}$ , these contracted algebras can be described as [34]

$$t_{a(N+1-a)} \odot (so_{\kappa_1, \dots, \kappa_{a-1}}(p, q) \oplus so_{\kappa_{a+1}, \dots, \kappa_N}(p', q')), \quad p+q = a, \quad p'+q' = N+1-a.$$

- The fully contracted case in the CK family corresponds to setting all  $\kappa_a = 0$ . This is the so called flag algebra  $so_{0, \dots, 0}(N+1) \equiv i \dots iso(1)$  [24] such that  $iso(1) \equiv \mathbb{R}$ .

We recall that the kinematical algebras associated to different models of spacetimes of constant curvature [35, 36] also belong to these CK algebras [32, 37] and they will be described in subsection 2.3.

## 2.2 Symmetrical homogeneous CK spaces

If we now consider the CK group  $SO_\kappa(N+1)$  with Lie algebra  $so_\kappa(N+1)$  we find that each Lie subalgebra  $h^{(m)}$  (2.8) generates a subgroup  $H^{(m)}$  leading to the homogeneous coset space denoted by:

$$\mathcal{S}^{(m)} \equiv SO_\kappa(N+1) / (SO_{\kappa_1, \dots, \kappa_{m-1}}(m) \otimes SO_{\kappa_{m+1}, \dots, \kappa_N}(N+1-m)). \quad (2.10)$$

The *dimension* of  $\mathcal{S}^{(m)}$  is that of  $p^{(m)}$  (see (2.9)) which is identified with the tangent space to  $\mathcal{S}^{(m)}$  at the origin:

$$\dim(\mathcal{S}^{(m)}) = m(N+1-m). \quad (2.11)$$

Then  $\mathcal{S}^{(m)}$  is a symmetrical homogeneous space (associated to the involution (2.2)), and there are  $N$  such symmetrical homogeneous spaces  $\mathcal{S}^{(m)}$  ( $m = 1, \dots, N$ ) for each CK group  $SO_\kappa(N+1)$ .

Notice that although some Lie algebras in the CK family  $so_\kappa(N+1)$  are isomorphic, their corresponding sets of  $N$  homogeneous spaces are different, as these are determined not only by  $so_\kappa(N+1)$ , but also by the subalgebra which will play the role of isotropy subalgebra of each individual coset space. Furthermore, these  $N$  spaces are not completely unrelated, and it is possible to reformulate all properties of any given space in terms of any other one. In particular,  $\mathcal{S}^{(2)}, \dots, \mathcal{S}^{(N)}$  are usually interpreted in terms of  $\mathcal{S}^{(1)}$ , which covers the classical Riemannian spaces and spacetimes of constant curvature; such an interpretation lies in the fact that the subgroups  $H^{(m)}$  ( $m = 1, 2, \dots, N$ ) are identified with the isotropy subgroups of a point ( $m = 1$ ), a line ( $m = 2$ ), ..., a hyperplane ( $m = N$ ) in  $\mathcal{S}^{(1)}$ . Hence, if  $\mathcal{S}^{(1)}$  is taken as *the* space, its elements are called *points*,  $\mathcal{S}^{(2)}$  is the space of all lines in  $\mathcal{S}^{(1)}$ ,  $\mathcal{S}^{(3)}$  is the space of all 2-planes in  $\mathcal{S}^{(1)}$ , etc.

Table 1: Isotropy subgroup, sectional curvature, dimension and rank of the set of  $N$  symmetrical homogeneous spaces  $\mathcal{S}^{(m)} \equiv SO_\kappa(N+1)/H^{(m)}$ .

Isotropy subgroup	Curv.	Dimension	Rank
$H^{(1)} = SO_{\kappa_2, \dots, \kappa_N}(N)$	$\kappa_1$	$N$	1
$H^{(2)} = SO_{\kappa_1}(2) \otimes SO_{\kappa_3, \dots, \kappa_N}(N-1)$	$\kappa_2$	$2(N-1)$	2
$H^{(3)} = SO_{\kappa_1, \kappa_2}(3) \otimes SO_{\kappa_4, \dots, \kappa_N}(N-2)$	$\kappa_3$	$3(N-2)$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H^{(m)} = SO_{\kappa_1, \dots, \kappa_{m-1}}(m) \otimes SO_{\kappa_{m+1}, \dots, \kappa_N}(N+1-m)$	$\kappa_m$	$m(N+1-m)$	$\min(m, N+1-m)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$H^{(N-2)} = SO_{\kappa_1, \dots, \kappa_{N-3}}(N-2) \otimes SO_{\kappa_{N-1}, \kappa_N}(3)$	$\kappa_{N-2}$	$(N-2)3$	3
$H^{(N-1)} = SO_{\kappa_1, \dots, \kappa_{N-2}}(N-1) \otimes SO_{\kappa_N}(2)$	$\kappa_{N-1}$	$(N-1)2$	2
$H^{(N)} = SO_{\kappa_1, \dots, \kappa_{N-1}}(N)$	$\kappa_N$	$N$	1

We define the *rank* of the CK space  $\mathcal{S}^{(m)}$  as the number of independent invariants under the action of the CK group for each generic pair of elements in  $\mathcal{S}^{(m)}$  (see [38] for the Euclidean case); such a number turns out to be the same for all  $\mathcal{S}^{(m)}$ , so it does not depend on the values of  $\kappa$ :

$$\text{rank}(\mathcal{S}^{(m)}) = \min(m, N+1-m). \quad (2.12)$$

Thus,  $\mathcal{S}^{(1)}$  has a single invariant (the ordinary distance) associated to each pair of points;  $\mathcal{S}^{(2)}$  has two invariants for each pair of lines (a “stationary angle” and a “distance” between the two lines), and, in general,  $\mathcal{S}^{(m)}$  has (2.12) invariants for a pair of  $(m - 1)$ -planes (these are called collectively “stationary angles”; the last of these is a single “stationary distance”).

The sectional *curvature* of  $\mathcal{S}^{(m)}$  turns out to be *essentially* constant and equal to  $\kappa_m$  (warning: we do not enter here into the details required to precise the “essentially”; let us only mention that in a space of rank equal to  $r$  there are flat  $r$ -dimensional subspaces; as the sectional curvature must vanish along any plane direction contained in such a subspace it cannot, of course, be constant in the ordinary sense; however the statement is meaningful with the due qualifications). We display in table 1 all these results concerning  $\mathcal{S}^{(m)}$ .

So far, we have interpreted each (graded) contraction parameter  $\kappa_m$ , appearing in an “abstract” form in the commutation relations of  $so_\kappa(N + 1)$  (2.3), as the constant curvature of the associated symmetry homogeneous space  $\mathcal{S}^{(m)} = SO_\kappa(N + 1)/H^{(m)}$ . Then when the contraction process is read in the reverse way, as a classical deformation one, we find that to introduce a non-zero constant  $\kappa_m$  in the Lie brackets (2.3) geometrically corresponds to the obtention of a curved space  $\mathcal{S}^{(m)}$  with  $\kappa_m \neq 0$  from an initial flat one with  $\kappa_m = 0$ ; notice that the “flat” and “non-relativistic” contraction/deformation examples commented in the introduction are recovered as two very particular cases within this framework for  $\kappa = (0, +\dots, +) \leftrightarrow (+1/R^2, +\dots, +)$  and  $(0, 0, +\dots, +) \leftrightarrow (0, -1/c^2, +\dots, +)$ , respectively. In this sense, we recall that the usual approach to Lie algebra deformations [6, 17] introduce non-zero structure constants in a given Lie algebra leading to a “less” Abelian one by applying cohomology techniques (see, e.g., [39] for a complete description of Galilean deformations). Thus by starting from the flag algebra  $so_{0,\dots,0}(N + 1)$  one could reach the simple ones  $so_\kappa(N + 1)$  with all  $\kappa_m \neq 0$ . Nevertheless such an algebraic procedure does not focus on the underlying homogeneous spaces. In contrast, an alternative deformation formalism makes use of universal enveloping algebras and of their associated homogeneous spaces [7, 18], in such a manner that the deformed generators are written as elements of the universal enveloping algebra to be deformed. We omit here the details, but we remark that this approach is directly related with the CK scheme of homogeneous spaces and, furthermore, this suggests some kind of relationship with quantum algebras, as these are also constructed within universal enveloping algebras; in fact we will establish such a connection in section 3.

### 2.3 Riemannian and (non-)relativistic spaces of constant curvature

From now on we assume that  $\kappa_3 = \dots = \kappa_N = +1$  and consider the rank-1 ND space

$$\mathcal{S}^{(1)} = SO_{\kappa_1, \kappa_2, +, \dots, +}(N + 1)/SO_{\kappa_2, +, \dots, +}(N) \equiv SO_{\kappa_1, \kappa_2}(N + 1)/SO_{\kappa_2}(N) \equiv \mathbb{S}_{[\kappa_1]\kappa_2}^N,$$

with sectional curvature  $\kappa_1$  and metric with signature determined by  $\kappa_2$  through the matrix  $\text{diag}(+1, \kappa_2, \dots, \kappa_2)$ . Hence we shall deal with the following nine well known spaces of constant curvature:

- When  $\kappa_2 > 0$ ,  $\mathbb{S}_{[\kappa_1]+}^N$  covers the three classical Riemannian spaces:

$$\begin{aligned} \text{Spherical:} \quad & \mathbb{S}_{[+] +}^N \equiv \mathbf{S}^N = SO(N+1)/SO(N). \\ \text{Euclidean:} \quad & \mathbb{S}_{[0] +}^N \equiv \mathbf{E}^N = ISO(N)/SO(N). \\ \text{Hyperbolic:} \quad & \mathbb{S}_{[-] +}^N \equiv \mathbf{H}^N = SO(N,1)/SO(N). \end{aligned}$$

Their curvature can be written as  $\kappa_1 = \pm 1/R^2$  where  $R$  is the radius of the space ( $R \rightarrow \infty$  for the Euclidean case).

- When  $\kappa_2 < 0$  we get a Lorentzian metric corresponding to relativistic spacetimes [35]:

$$\begin{aligned} \text{Anti-de Sitter:} \quad & \mathbb{S}_{[+] -}^N \equiv \mathbf{AdS}^{(N-1)+1} = SO(N-1,2)/SO(N-1,1). \\ \text{Minkowskian:} \quad & \mathbb{S}_{[0] -}^N \equiv \mathbf{M}^{(N-1)+1} = ISO(N-1,1)/SO(N-1,1). \\ \text{de Sitter:} \quad & \mathbb{S}_{[-] -}^N \equiv \mathbf{dS}^{(N-1)+1} = SO(N,1)/SO(N-1,1). \end{aligned}$$

The two contraction parameters can be expressed as  $\kappa_1 = \pm 1/\tau^2$ , where  $\tau$  is the (time) universe radius, and  $\kappa_2 = -1/c^2$ , where  $c$  is the speed of light.

- The contraction  $\kappa_2 = 0$  ( $c \rightarrow \infty$ ) gives rise to the non-relativistic spacetimes with a degenerate metric [35]:

$$\begin{aligned} \text{Oscillating NH:} \quad & \mathbb{S}_{[+] 0}^N \equiv \mathbf{NH}_+^{(N-1)+1} = T_{2N-2} \odot (SO(2) \otimes SO(N-1))/ISO(N-1). \\ \text{Galilean:} \quad & \mathbb{S}_{[0] 0}^N \equiv \mathbf{G}^{(N-1)+1} = IISO(N-1)/ISO(N-1). \\ \text{Expanding NH:} \quad & \mathbb{S}_{[-] 0}^N \equiv \mathbf{NH}_-^{(N-1)+1} = T_{2N-2} \odot (SO(1,1) \otimes SO(N-1))/ISO(N-1). \end{aligned}$$

In what follows we construct an explicit model of the space  $\mathbb{S}_{[\kappa_1]\kappa_2}^N$  in terms of  $(N+1)$  ambient coordinates and of  $N$  intrinsic quantities. The CK algebra  $so_{\kappa_1, \kappa_2}(N+1)$  has a vector representation given by the following  $(N+1) \times (N+1)$  real matrices fulfilling (2.3):

$$J_{ab} = -\kappa_{ab}e_{ab} + e_{ba}, \quad (2.13)$$

where  $e_{ab}$  is the matrix with entries  $(e_{ab})_m^l = \delta_a^l \delta_b^m$ . In this realization, any element  $X \in so_{\kappa_1, \kappa_2}(N+1)$  satisfies the equation:

$$X^T \mathbb{I}_\kappa + \mathbb{I}_\kappa X = 0, \quad \mathbb{I}_\kappa = \text{diag}(+1, \kappa_1, \kappa_1 \kappa_2, \dots, \kappa_1 \kappa_2), \quad (2.14)$$

where  $X^T$  is the transpose matrix of  $X$ . Hence any element  $G \in SO_{\kappa_1, \kappa_2}(N+1)$  verifies  $G^T \mathbb{I}_\kappa G = \mathbb{I}_\kappa$  and  $SO_{\kappa_1, \kappa_2}(N+1)$  is a group of isometries of  $\mathbb{I}_\kappa$  acting on a linear ambient space  $\mathbb{R}^{N+1} = (x_0, x_1, \dots, x_N)$  through matrix multiplication. The origin  $\mathcal{O}$  in  $\mathbb{S}_{[\kappa_1]\kappa_2}^N$  has  $(N+1)$  ambient coordinates  $\mathcal{O} = (1, 0, \dots, 0)$  which is invariant under the subgroup  $H^{(1)} = SO_{\kappa_2}(N)$ . The orbit of  $\mathcal{O}$  corresponds to  $\mathbb{S}_{[\kappa_1]\kappa_2}^N$  which is contained in the “sphere” determined by  $\mathbb{I}_\kappa$ :

$$\Sigma \equiv x_0^2 + \kappa_1 x_1^2 + \kappa_1 \kappa_2 \sum_{j=2}^N x_j^2 = 1. \quad (2.15)$$

The CK metric on  $\mathbb{S}_{[\kappa_1]\kappa_2}^N$  follows from the flat ambient metric in  $\mathbb{R}^{N+1}$  in the form

$$ds_{\text{CK}}^2 = \frac{1}{\kappa_1} \left( dx_0^2 + \kappa_1 dx_1^2 + \kappa_1 \kappa_2 \sum_{j=2}^N dx_j^2 \right) \Big|_\Sigma. \quad (2.16)$$



Next we parametrize the  $(N + 1)$  ambient coordinates  $\mathbf{x}$  of a generic point  $\mathcal{P}$  in terms of  $N$  intrinsic quantities  $(r, \theta, \phi_3, \dots, \phi_N)$  called *geodesic polar coordinates* [40] on  $\mathbb{S}_{[\kappa_1]\kappa_2}^N$  through the following action of  $N$  one-parametric subgroups of  $SO_{\kappa_1, \kappa_2}(N + 1)$  on  $\mathcal{O}$ :

$$\mathbf{x} = \exp(\phi_N J_{N-1 N}) \exp(\phi_{N-1} J_{N-2 N-1}) \dots \exp(\phi_3 J_{23}) \exp(\theta J_{12}) \exp(r J_{01}) \mathcal{O}, \quad (2.17)$$

which yields

$$\begin{aligned} x_0 &= C_{\kappa_1}(r), \\ x_1 &= S_{\kappa_1}(r) C_{\kappa_2}(\theta), \\ x_i &= S_{\kappa_1}(r) S_{\kappa_2}(\theta) \prod_{s=3}^i \sin \phi_s \cos \phi_{i+1}, \\ x_N &= S_{\kappa_1}(r) S_{\kappa_2}(\theta) \prod_{s=3}^N \sin \phi_s, \end{aligned} \quad (2.18)$$

where  $i = 2, \dots, N - 1$  and any product  $\prod_s^i$  where  $s > i$  is assumed to be equal to 1. The  $\kappa$ -trigonometric functions  $C_\kappa(x)$  and  $S_\kappa(x)$  are defined by [40] (here for  $\kappa \in \{\kappa_1, \kappa_2\}$ ):

$$C_\kappa(x) = \begin{cases} \cos \sqrt{\kappa} x, & \kappa > 0, \\ 1, & \kappa = 0, \\ \cosh \sqrt{-\kappa} x, & \kappa < 0. \end{cases} \quad S_\kappa(x) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x, & \kappa > 0, \\ x, & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x, & \kappa < 0. \end{cases} \quad (2.19)$$

The (physical) geometrical role of these coordinates is as follows. Let us consider a (time-like) geodesic  $l_1$  and other  $(N - 1)$  (space-like) geodesics  $l_j$  ( $j = 2, \dots, N$ ) in  $\mathbb{S}_{[\kappa_1]\kappa_2}^N$  which are orthogonal at the origin  $\mathcal{O}$  in such a manner that each translation generator  $J_{0i}$  moves  $\mathcal{O}$  along  $l_i$ . Then,

- The radial coordinate  $r$  is the distance between the point  $\mathcal{P}$  and the origin  $\mathcal{O}$  measured along the geodesic  $l$  that joins both points. In the Riemannian spaces with  $\kappa_1 = \pm 1/R^2$ ,  $r$  has dimensions of *length*,  $[r] = [R]$ ; notice that the dimensionless coordinate  $r/R$  is usually taken instead of  $r$  and  $r/R$  is an ordinary angle [41]. In the spacetimes with  $\kappa_1 = \pm 1/\tau^2$ ,  $r$  has dimensions of a time-like length,  $[r] = [\tau]$ .
- The coordinate  $\theta$  is an ordinary angle in the three Riemannian spaces ( $\kappa_2 = +1$ ), while it corresponds to a rapidity in the spacetimes ( $\kappa_2 = -1/c^2$ ) with dimensions  $[\theta] = [c]$ . For the nine spaces,  $\theta$  parametrizes the orientation of  $l$  with respect to the basic (time-like) geodesic  $l_1$ .
- The remaining  $(N - 2)$  coordinates  $\phi_3, \phi_4, \dots, \phi_N$  are ordinary angles for the nine spaces and correspond to the polar angles of  $l$  relative to the reference flag at the origin  $\mathcal{O}$  spanned by  $\{l_1, l_2\}, \{l_1, l_2, l_3\}, \dots, \{l_1, \dots, l_{N-1}\}$ , respectively.

In the three Riemannian cases  $(r, \theta, \phi_3, \dots, \phi_N)$  parametrize the complete space, while in the relativistic spacetimes these only cover the time-like region limited by the light-cone on which  $\theta \rightarrow \infty$ . The flat contraction  $\kappa_1 = 0$  gives rise to the usual spherical coordinates in the Euclidean space (with  $\kappa_2 = +1$ ).

By introducing (2.18) in (2.16), we obtain the CK metric in  $\mathbb{S}_{[\kappa_1]\kappa_2}^N$  expressed in geodesic polar coordinates:

$$ds_{\text{CK}}^2 = dr^2 + \kappa_2 S_{\kappa_1}^2(r) \left\{ d\theta^2 + S_{\kappa_2}^2(\theta) \sum_{i=3}^N \left( \prod_{s=3}^{i-1} \sin^2 \phi_s \right) d\phi_i^2 \right\}. \quad (2.20)$$

Table 2: Metric, sectional and scalar curvatures of the nine 3D CK spaces  $SO_{\kappa_1, \kappa_2}(4)/SO_{\kappa_2}(3)$  expressed in geodesic polar coordinates according to  $\kappa_1, \kappa_2 \in \{\pm 1, 0\}$ .

<ul style="list-style-type: none"> <li>• Sphere <math>\mathbf{S}^3</math>  <math>SO(4)/SO(3)</math>  <math>(\kappa_1, \kappa_2) = (+1, +1)</math>  <math>ds_{\text{CK}}^2 = dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2)</math>  <math>K_{ij} = +1 \quad K = +6</math></li> </ul>	<ul style="list-style-type: none"> <li>• Euclidean <math>\mathbf{E}^3</math>  <math>ISO(3)/SO(3)</math>  <math>(\kappa_1, \kappa_2) = (0, +1)</math>  <math>ds_{\text{CK}}^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)</math>  <math>K_{ij} = 0 \quad K = 0</math></li> </ul>	<ul style="list-style-type: none"> <li>• Hyperbolic <math>\mathbf{H}^3</math>  <math>SO(3, 1)/SO(3)</math>  <math>(\kappa_1, \kappa_2) = (-1, +1)</math>  <math>ds_{\text{CK}}^2 = dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2)</math>  <math>K_{ij} = -1 \quad K = -6</math></li> </ul>
<ul style="list-style-type: none"> <li>• Oscillating NH <math>\mathbf{NH}_+^{2+1}</math>  <math>T_4 \odot (SO(2) \otimes SO(2))/ISO(2)</math>  <math>(\kappa_1, \kappa_2) = (+1, 0)</math>  <math>ds_{\text{CK}}^2 = dr^2</math>  <math>K_{ij} = +1 \quad K = +6</math></li> </ul>	<ul style="list-style-type: none"> <li>• Galilean <math>\mathbf{G}^{2+1}</math>  <math>IISO(2)/ISO(2)</math>  <math>(\kappa_1, \kappa_2) = (0, 0)</math>  <math>ds_{\text{CK}}^2 = dr^2</math>  <math>K_{ij} = 0 \quad K = 0</math></li> </ul>	<ul style="list-style-type: none"> <li>• Expanding NH <math>\mathbf{NH}_-^{2+1}</math>  <math>T_4 \odot (SO(1, 1) \otimes SO(2))/ISO(2)</math>  <math>(\kappa_1, \kappa_2) = (-1, 0)</math>  <math>ds_{\text{CK}}^2 = dr^2</math>  <math>K_{ij} = -1 \quad K = -6</math></li> </ul>
<ul style="list-style-type: none"> <li>• Anti-de Sitter <math>\mathbf{AdS}^{2+1}</math>  <math>SO(2, 2)/SO(2, 1)</math>  <math>(\kappa_1, \kappa_2) = (+1, -1)</math>  <math>ds_{\text{CK}}^2 = dr^2 - \sin^2 r (d\theta^2 + \sinh^2 \theta d\phi^2)</math>  <math>K_{ij} = +1 \quad K = +6</math></li> </ul>	<ul style="list-style-type: none"> <li>• Minkowskian <math>\mathbf{M}^{2+1}</math>  <math>ISO(2, 1)/SO(2, 1)</math>  <math>(\kappa_1, \kappa_2) = (0, -1)</math>  <math>ds_{\text{CK}}^2 = dr^2 - r^2 (d\theta^2 + \sinh^2 \theta d\phi^2)</math>  <math>K_{ij} = 0 \quad K = 0</math></li> </ul>	<ul style="list-style-type: none"> <li>• de Sitter <math>\mathbf{dS}^{2+1}</math>  <math>SO(3, 1)/SO(2, 1)</math>  <math>(\kappa_1, \kappa_2) = (-1, -1)</math>  <math>ds_{\text{CK}}^2 = dr^2 - \sinh^2 r (d\theta^2 + \sinh^2 \theta d\phi^2)</math>  <math>K_{ij} = -1 \quad K = -6</math></li> </ul>

The sectional  $K_{ij}$  and the scalar  $K$  curvatures are  $K_{ij} = \kappa_1$  and  $K = N(N-1)\kappa_1$ ; in this rank-one case, the sectional curvature along any 2D-direction equals precisely  $\kappa_1$  and hence is actually constant in the literal sense.

As an example, which is also necessary for our further development in relation with quantum deformations, we display in table 2 these results for  $N = 3$ .

### 3 Contraction, curvature and quantum algebras

#### 3.1 A quantum deformation of $sl(2, \mathbb{R})$

Let us consider the algebra of formal power series in a *real* deformation parameter  $z$  ( $q = e^z$ ) with coefficients in  $U(sl(2, \mathbb{R}))$ . If this algebra is endowed with a (deformed) Hopf structure [8] we get the so called *non-standard quantum deformation* of  $U(sl(2, \mathbb{R}))$ , here denoted by  $U_z(sl(2, \mathbb{R})) \equiv sl_z(2)$ . The Poisson analogue of this quantum algebra is given by the following deformed Poisson brackets and coproduct map  $\Delta$  [19, 29]:

$$\{J_3, J_+\} = 2J_+ \cosh zJ_-, \quad \{J_3, J_-\} = -2 \frac{\sinh zJ_-}{z}, \quad \{J_-, J_+\} = 4J_3, \quad (3.1)$$

$$\begin{aligned} \Delta(J_-) &= J_- \otimes 1 + 1 \otimes J_-, \\ \Delta(J_l) &= J_l \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_l, \quad l = +, 3. \end{aligned} \quad (3.2)$$

The deformed Casimir function for (3.1) reads

$$\mathcal{C} = \frac{\sinh zJ_-}{z} J_+ - J_3^2. \quad (3.3)$$

A one-particle symplectic realization of (3.1) is given by [19]

$$J_-^{(1)} = q_1^2, \quad J_+^{(1)} = \frac{\sinh z q_1^2}{z q_1^2} p_1^2, \quad J_3^{(1)} = \frac{\sinh z q_1^2}{z q_1^2} q_1 p_1, \quad (3.4)$$

so that  $\mathcal{C}^{(1)} = 0$ . By starting from (3.4), the coproduct (3.2) determines the corresponding two-particle realization of (3.1), which is defined on  $sl_z(2) \otimes sl_z(2)$ :

$$\begin{aligned} J_-^{(2)} &= q_1^2 + q_2^2, & J_+^{(2)} &= \left( \frac{\sinh z q_1^2}{z q_1^2} e^{z q_2^2} \right) p_1^2 + \left( \frac{\sinh z q_2^2}{z q_2^2} e^{-z q_1^2} \right) p_2^2, \\ J_3^{(2)} &= \left( \frac{\sinh z q_1^2}{z q_1^2} e^{z q_2^2} \right) q_1 p_1 + \left( \frac{\sinh z q_2^2}{z q_2^2} e^{-z q_1^2} \right) q_2 p_2. \end{aligned} \quad (3.5)$$

Then the two-particle Casimir is given by

$$\begin{aligned} \mathcal{C}^{(2)} &= \frac{\sinh z J_-^{(2)}}{z} J_+^{(2)} - \left( J_3^{(2)} \right)^2 \\ &= \left( \frac{\sinh z q_1^2}{z q_1^2} \frac{\sinh z q_2^2}{z q_2^2} e^{-z q_1^2} e^{z q_2^2} \right) (q_1 p_2 - q_2 p_1)^2. \end{aligned} \quad (3.6)$$

This procedure can be iterated to arbitrary  $N$ . In particular the 3-sites coproduct,  $\Delta^{(3)} = (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ , gives rise to a three-particle symplectic realization of (3.1) defined on  $sl_z(2) \otimes sl_z(2) \otimes sl_z(2)$ :

$$\begin{aligned} J_-^{(3)} &= q_1^2 + q_2^2 + q_3^2 \equiv \mathbf{q}^2, \\ J_3^{(3)} &= \left( \frac{\sinh z q_1^2}{z q_1^2} e^{z q_2^2} e^{z q_3^2} \right) q_1 p_1 + \left( \frac{\sinh z q_2^2}{z q_2^2} e^{-z q_1^2} e^{z q_3^2} \right) q_2 p_2 \\ &\quad + \left( \frac{\sinh z q_3^2}{z q_3^2} e^{-z q_1^2} e^{-z q_2^2} \right) q_3 p_3, \\ J_+^{(3)} &= \left( \frac{\sinh z q_1^2}{z q_1^2} e^{z q_2^2} e^{z q_3^2} \right) p_1^2 + \left( \frac{\sinh z q_2^2}{z q_2^2} e^{-z q_1^2} e^{z q_3^2} \right) p_2^2 + \left( \frac{\sinh z q_3^2}{z q_3^2} e^{-z q_1^2} e^{-z q_2^2} \right) p_3^2. \end{aligned} \quad (3.7)$$

It is immediate to check that these three functions fulfil the commutation rules (3.1) with respect to the canonical Poisson bracket

$$\{f, g\} = \sum_{i=1}^3 \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right). \quad (3.8)$$

Likewise, the three-particle Casimir function  $\mathcal{C}^{(3)}$  can be straightforwardly obtained.

In this way we have obtained a (three-particle) *quantum deformation*, in a generic Hopf algebra framework, of the  $sl(2, \mathbb{R})$  Lie–Poisson algebra, understood as a more general structure that depends on the “additional” quantum deformation parameter  $z$ . This means that under the *classical limit*  $z \rightarrow 0$  (or  $q \rightarrow 1$ ) the non-deformed Lie–Poisson brackets, Casimir, primitive coproduct ( $\Delta(X) = X \otimes 1 + 1 \otimes X$ ) and symplectic realization of  $sl(2, \mathbb{R})$  are recovered, the latter being  $J_-^{(3)} = \mathbf{q}^2$ ,  $J_+^{(3)} = \mathbf{p}^2 = \sum_{i=1}^3 p_i^2$ ,  $J_3^{(3)} = \mathbf{q} \cdot \mathbf{p} = \sum_{i=1}^3 q_i p_i$ .

In the sequel we will show that this quantum deformation can be interpreted as an algebraic/geometric tool that introduces a *non-constant curvature* in a formerly *flat* 3D Euclidean space  $\mathbf{E}^3$ , in such a manner that the deformation parameter  $z$  governs the (non-constant) curvature of the underlying space. The number of copies of  $sl_z(2, \mathbb{R})$  is just the dimensionality of the space, and further iterations of the coproduct map would lead to an ND construction.

### 3.2 Riemannian and (non-)relativistic spaces of non-constant curvature

An infinite family of 3D *free* (kinetic energy) Hamiltonians  $\mathcal{T}$  [20] can be constructed in terms of the generators (3.7) in the form:

$$\mathcal{T} = \frac{1}{2} J_+^{(3)} f(z J_-^{(3)}), \quad (3.9)$$

where  $f$  is an arbitrary smooth function such that  $\lim_{z \rightarrow 0} f(z J_-^{(3)}) = 1$ ; in this way  $\lim_{z \rightarrow 0} \mathcal{T} = \frac{1}{2} \mathbf{p}^2$  gives the usual kinetic energy on  $\mathbf{E}^3$ . Thus by writing (3.9) as the free Lagrangian,

$$\mathcal{T} = \frac{1}{2} \left( \frac{z q_1^2}{\sinh z q_1^2} e^{-z q_2^2} e^{-z q_3^2} \dot{q}_1^2 + \frac{z q_2^2}{\sinh z q_2^2} e^{z q_1^2} e^{-z q_3^2} \dot{q}_2^2 + \frac{z q_3^2}{\sinh z q_3^2} e^{z q_1^2} e^{z q_2^2} \dot{q}_3^2 \right) f(z \mathbf{q}^2), \quad (3.10)$$

we obtain the geodesic flow on the 3D space whose definite positive metric is given by

$$ds^2 = \left( \frac{2z q_1^2}{\sinh z q_1^2} e^{-z q_2^2} e^{-z q_3^2} dq_1^2 + \frac{2z q_2^2}{\sinh z q_2^2} e^{z q_1^2} e^{-z q_3^2} dq_2^2 + \frac{2z q_3^2}{\sinh z q_3^2} e^{z q_1^2} e^{z q_2^2} dq_3^2 \right) \frac{1}{f(z \mathbf{q}^2)}. \quad (3.11)$$

If one computes the corresponding sectional  $K_{ij}$  and scalar  $K$  curvatures associated to the metric (3.11) one finds that, in general, these are non-constant; the latter turns out to be

$$K(x) = z \left( 6f'(x) \cosh x + \left( 4f''(x) - 5f(x) - 5f'^2(x)/f(x) \right) \sinh x \right), \quad (3.12)$$

where  $x \equiv z J_-^{(3)} = z \mathbf{q}^2$ ,  $f'(x) = \frac{df(x)}{dx}$  and  $f''(x) = \frac{d^2f(x)}{dx^2}$ . This, in turn, means that the deformed coalgebra process can be understood as the introduction of a non-constant curvature on a formerly flat space  $\mathbf{E}^3$ . Hence the non-deformed or “classical” limit  $z \rightarrow 0$  can then be identified with a proper *flat contraction*, under the which, the metric (3.11) reduces to the 3D Euclidean one in Cartesian coordinates,  $ds^2 = \sum_{i=1}^3 dq_i^2$ , and the scalar curvature (3.12) vanishes for any choice of the arbitrary function  $f$  (which always reduces to 1).

Furthermore the metric (3.11) can be rewritten in order to give rise to curved spaces of pseudo- and semi-Riemannian type as well (with Lorentzian and degenerate metrics), which can be thought as non-constant curvature deformations of the CK spaces described in section 2.3. Explicitly, we apply the following change of coordinates from  $\mathbf{q}$  to the polar-type ones  $(r, \theta, \phi)$  (compare to (2.18) for  $N = 3$ ):

$$\begin{aligned} \cos^2(\lambda_1 r) &= e^{-2z \mathbf{q}^2}, \\ \tan^2(\lambda_1 r) \cos^2(\lambda_2 \theta) &= e^{2z q_1^2} e^{2z q_2^2} (e^{2z q_3^2} - 1), \\ \tan^2(\lambda_1 r) \sin^2(\lambda_2 \theta) \cos^2 \phi &= e^{2z q_1^2} (e^{2z q_2^2} - 1), \\ \tan^2(\lambda_1 r) \sin^2(\lambda_2 \theta) \sin^2 \phi &= e^{2z q_1^2} - 1, \end{aligned} \quad (3.13)$$

where we have denoted  $z = \lambda_1^2$  and we have introduced an additional parameter  $\lambda_2$  which can be either a real or a pure imaginary number [19]. In this way, we find that the initial Riemannian metric (3.11) is mapped into

$$\begin{aligned} ds^2 &= \frac{1}{\cos(\lambda_1 r) g(\lambda_1 r)} \left( dr^2 + \lambda_2^2 \frac{\sin^2(\lambda_1 r)}{\lambda_1^2} \left( d\theta^2 + \frac{\sin^2(\lambda_2 \theta)}{\lambda_2^2} d\phi^2 \right) \right) \\ &= \frac{1}{\cos(\lambda_1 r) g(\lambda_1 r)} ds_{\text{CK}}^2, \end{aligned} \quad (3.14)$$

where  $g(\lambda_1 r) \equiv f(z\mathbf{q}^2)$  is an arbitrary smooth function such that  $\lim_{\lambda_1 \rightarrow 0} g(\lambda_1 r) = 1$ . Thus we have obtained a family of metrics, parametrized by  $\lambda_1, \lambda_2$  and depending on the function  $g$ , which is just the metric of the 3D CK spaces of constant curvature  $ds_{\text{CK}}^2$  (2.20) multiplied by a “conformal” factor  $1/(\cos(\lambda_1 r)g(\lambda_1 r))$ , once we identify

$$\kappa_1 \equiv z = \lambda_1^2, \quad \kappa_2 \equiv \lambda_2^2, \quad \kappa_3 = +1. \quad (3.15)$$

Consequently,  $z$  plays a *threefold role* as a quantum deformation/contraction/curvature parameter, while  $\lambda_2$  is a (graded) classical contraction/signature parameter which allows us to deal, simultaneously, with Riemannian, Lorentzian and degenerate metrics.

In this new coordinates the scalar curvature  $K$  (3.12) reads

$$K(y) = 2z \cos y \left( \left( \frac{1 + 3 \cos^2 y}{\sin 2y} \right) g'(y) + g''(y) - \frac{5}{4} \frac{g'(y)^2}{g(y)} - \frac{5}{4} g(y) \tan^2 y \right), \quad (3.16)$$

where the radial variable  $y = \lambda_1 r$ . Then, according to the real values that  $z = \lambda_1^2$  and  $\lambda_2^2$  can take, we find that (3.14) comprises the following types of spaces:

- When  $z = \lambda_1^2 > 0$ , we obtain a family of 3D “deformed” spherical  $\mathbf{S}_z^3$  ( $\lambda_2^2 > 0$ ), oscillating NH  $\mathbf{NH}_{+,z}^{2+1}$  ( $\lambda_2 = 0$ ) and anti-de Sitter  $\mathbf{AdS}_z^{2+1}$  ( $\lambda_2^2 < 0$ ) spaces. For  $z = 1$  ( $\lambda_1 = 1$ ), their scalar curvature reads

$$K(r) = 2 \cos r \left( \left( \frac{1 + 3 \cos^2 r}{\sin 2r} \right) g'(r) + g''(r) - \frac{5}{4} \frac{g'(r)^2}{g(r)} - \frac{5}{4} g(r) \tan^2 r \right).$$

- When  $z = \lambda_1^2 = 0$  we recover the proper *flat*  $\mathbf{E}^3$  ( $\lambda_2^2 > 0$ ),  $\mathbf{G}^{2+1}$  ( $\lambda_2^2 = 0$ ) and  $\mathbf{M}^{2+1}$  ( $\lambda_2^2 < 0$ ) spaces of table 2, all of them with  $K_{ij} = K = 0$ . The underlying symmetry remains as a Lie–Poisson one (non-deformed).
- And when  $z = \lambda_1^2 < 0$ , we get a family of “deformed” 3D hyperbolic  $\mathbf{H}_z^3$  ( $\lambda_2^2 > 0$ ), expanding NH  $\mathbf{NH}_{-,z}^{2+1}$  ( $\lambda_2 = 0$ ) and de Sitter  $\mathbf{dS}_z^{2+1}$  ( $\lambda_2^2 < 0$ ) spaces, with scalar curvature for  $z = -1$  ( $\lambda_1 = i$ ) given by

$$K(r) = -2 \cosh r \left( \left( \frac{1 + 3 \cosh^2 r}{i \sinh 2r} \right) g'(ir) + g''(ir) - \frac{5}{4} \frac{g'(ir)^2}{g(ir)} + \frac{5}{4} g(ir) \tanh^2 r \right).$$

In order to illustrate explicitly the above results we consider the simplest example corresponding to set  $f(zJ_-^{(3)}) = f(z\mathbf{q}^2) = g(\lambda_1 r) \equiv 1$  in (3.11), that is,  $\mathcal{T} = \frac{1}{2}J_+^{(3)}$ . In this case the sectional  $K_{ij}$  and scalar  $K$  curvatures in the coordinates  $\mathbf{q}$  turn out to be [20]:

$$\begin{aligned} K_{12} &= \frac{z}{4} e^{-z\mathbf{q}^2} \left( 1 + e^{2zq_3^2} - 2e^{2z\mathbf{q}^2} \right), \\ K_{13} &= \frac{z}{4} e^{-z\mathbf{q}^2} \left( 2 - e^{2zq_3^2} + e^{2z(q_2^2+q_3^2)} - 2e^{2z\mathbf{q}^2} \right), \\ K_{23} &= \frac{z}{4} e^{-z\mathbf{q}^2} \left( 2 - e^{2z(q_2^2+q_3^2)} - 2e^{2z\mathbf{q}^2} \right), \\ K &= 2(K_{12} + K_{13} + K_{23}) = -5z \sinh(z\mathbf{q}^2). \end{aligned} \quad (3.17)$$

Table 3: Metric, sectional and scalar curvatures of six 3D spaces of non-constant curvature expressed in polar-type coordinates with  $z = \lambda_1^2 \in \{\pm 1\}$  and  $\lambda_2^2 \in \{\pm 1, 0\}$ . For the six cases the sectional curvature  $K_{23} = K_{1j}/2$  with  $j = 2, 3$ .

<ul style="list-style-type: none"> <li>• Deformed sphere <math>\mathbf{S}_z^3</math>  <math>z = +1; (\lambda_1, \lambda_2) = (1, 1)</math>  <math>ds^2 = \frac{1}{\cos r} (dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2))</math>  <math>K_{1j} = -\frac{\sin^2 r}{2 \cos r} \quad K = -\frac{5 \sin^2 r}{2 \cos r}</math></li> </ul>	<ul style="list-style-type: none"> <li>• Deformed hyperbolic <math>\mathbf{H}_z^3</math>  <math>z = -1; (\lambda_1, \lambda_2) = (i, 1)</math>  <math>ds^2 = \frac{1}{\cosh r} (dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2))</math>  <math>K_{1j} = -\frac{\sinh^2 r}{2 \cosh r} \quad K = -\frac{5 \sinh^2 r}{2 \cosh r}</math></li> </ul>
<ul style="list-style-type: none"> <li>• Deformed oscillating NH <math>\mathbf{NH}_{+,z}^{2+1}</math>  <math>z = +1; (\lambda_1, \lambda_2) = (1, 0)</math>  <math>ds^2 = \frac{1}{\cos r} dr^2</math>  <math>K_{1j} = -\frac{\sin^2 r}{2 \cos r} \quad K = -\frac{5 \sin^2 r}{2 \cos r}</math></li> </ul>	<ul style="list-style-type: none"> <li>• Deformed expanding NH <math>\mathbf{NH}_{-,z}^{2+1}</math>  <math>z = -1; (\lambda_1, \lambda_2) = (i, 0)</math>  <math>ds^2 = \frac{1}{\cosh r} dr^2</math>  <math>K_{1j} = -\frac{\sinh^2 r}{2 \cosh r} \quad K = -\frac{5 \sinh^2 r}{2 \cosh r}</math></li> </ul>
<ul style="list-style-type: none"> <li>• Deformed anti-de Sitter <math>\mathbf{AdS}_z^{2+1}</math>  <math>z = +1; (\lambda_1, \lambda_2) = (1, i)</math>  <math>ds^2 = \frac{1}{\cos r} (dr^2 - \sin^2 r (d\theta^2 + \sinh^2 \theta d\phi^2))</math>  <math>K_{1j} = -\frac{\sin^2 r}{2 \cos r} \quad K = -\frac{5 \sin^2 r}{2 \cos r}</math></li> </ul>	<ul style="list-style-type: none"> <li>• Deformed de Sitter <math>\mathbf{dS}_z^{2+1}</math>  <math>z = -1; (\lambda_1, \lambda_2) = (i, i)</math>  <math>ds^2 = \frac{1}{\cosh r} (dr^2 - \sinh^2 r (d\theta^2 + \sinh^2 \theta d\phi^2))</math>  <math>K_{1j} = -\frac{\sinh^2 r}{2 \cosh r} \quad K = -\frac{5 \sinh^2 r}{2 \cosh r}</math></li> </ul>

In the polar-type coordinates with metric  $ds^2 = ds_{\text{CK}}^2 / \cos(\lambda_1 r)$  these curvatures read

$$K_{12} = K_{13} = -\frac{1}{2} \lambda_1^2 \frac{\sin^2(\lambda_1 r)}{\cos(\lambda_1 r)}, \quad K_{23} = \frac{1}{2} K_{12}, \quad K = -\frac{5}{2} \lambda_1^2 \frac{\sin^2(\lambda_1 r)}{\cos(\lambda_1 r)}. \quad (3.18)$$

We display in table 3 the six particular “deformed” spaces (with non-constant curvature) arising for  $g = 1$ . We omit the Euclidean, Galilean and Minkowskian spaces, with  $z = 0$ , as these remain flat/non-deformed as given in table 2.

We remark that, in general, other choices for the geodesic motion Hamiltonian (3.9) (with  $f \neq 1$ ) give rise to more complicated spaces of non-constant curvature. We also stress that the nine CK spaces of table 2 can also be directly recovered from an  $sl_z(2)$ -coalgebra symmetry by setting  $g(\lambda_1 r) = 1/\cos(\lambda_1 r)$  ( $f(z\mathbf{q}^2) = e^{z\mathbf{q}^2}$ ), that is,  $\mathcal{T} = \frac{1}{2} J_+^{(3)} e^{zJ_-^{(3)}}$ . This is a very singular case amongst the whole family of curved spaces determined by the metric (3.14) since in this case all the curvatures are *constant*:  $K_{ij} = z \equiv \kappa_1$  and  $K = 6z \equiv 6\kappa_1$ . Once again, the role of the deformation parameter  $z$  as a curvature becomes striking.

## 4 Concluding remarks

The aim of this paper is to illustrate how a curvature can be understood either as a contraction parameter or as a quantum deformation one. This is explicitly achieved by constructing, respectively, a family of symmetrical homogeneous CK spaces from a theoretical Lie group approach and some non-constant curved spaces from a quantum group one. We remark that although the CK algebras/spaces have been already described in

arbitrary dimension  $N$ , their quantum deformed counterpart has only been presented here for  $N = 3$ . We recall that the coalgebra procedure [20, 42] affords for the ND generalization of any 2D result which, in fact, comes from the coproduct of the quantum algebra (so covering all the expressions given in subsection 3.1), but a clear geometrical/physical interpretation of the non-constant curved spaces is not so straightforward. A deeper study of the ND coalgebra curved spaces is currently under investigation.

On the other hand, from a dynamical viewpoint, all the geodesic motions associated to the family of (quantum deformed) metrics (3.14) are, in general, superintegrable since they are endowed with *three* functionally independent integrals of motion, besides the free Hamiltonian. Such integrals come from the 2- and 3-particle Casimirs, and can be explicitly constructed. Nevertheless, by using the coalgebra approach there is always a constant of the motion left in order to ensure maximal superintegrability (this is a completely general fact [20]). Such a family of geodesic motion Hamiltonians associated to (3.14), in coordinates  $(r, \theta, \phi)$  and canonical conjugated momenta  $(p_r, p_\theta, p_\phi)$ , reads

$$T = \frac{1}{2} \cos(\lambda_1 r) g(\lambda_1 r) \left( p_r^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)} \left( p_\theta^2 + \frac{\lambda_2^2}{\sin^2(\lambda_2 \theta)} p_\phi^2 \right) \right), \quad (4.19)$$

where  $T = 2\mathcal{T}$  (3.9). In this respect, we also stress that different superintegrable potentials [20] on curved spaces with  $sl_z(2)$ -symmetry can be obtained by adding a potential term  $U(zJ_-)$  to  $T$  (4.19), since the superintegrability properties of the complete Hamiltonian  $H = T + U$  can be shown to be preserved due to the underlying (quantum) coalgebra symmetry. Moreover, for the particular CK metrics (with  $g(\lambda_1 r) = 1/\cos(\lambda_1 r)$ ) it is possible obtain the additional integral by Lie algebraic methods, so that the corresponding kinetic energy on the CK spaces is, as it is well known, maximally superintegrable. Finally we point out a fact worthy of consideration: although the underlying (deformed and CK) curved spaces are always well defined for any value of  $\lambda_1$  and  $\lambda_2$ , their corresponding metrics cannot be used in a dynamical picture for the Newtonian spaces with degenerate metrics since if  $\lambda_2 \rightarrow 0$ , then  $T \rightarrow \infty$ .

## Acknowledgments

This work was partially supported by the Ministerio de Educación y Ciencia (Spain, Projects FIS2004-07913 and MTM2005-09183), by the Junta de Castilla y León (Spain, Project VA013C05), and by the INFN-CICyT (Italy-Spain).

## References

- [1] Segal I.E. (1951). *Duke Math. J.* **18**, 221.
- [2] İnönü E., Wigner E.P. (1953). *Proc. Natl. Acad. Sci., USA* **39**, 510; *ibid*, (1954). **40**, 119.
- [3] Saletan E.J. (1961). *J. Math. Phys.* **2**, 1.
- [4] Weimar-Woods E. (1995). *J. Math. Phys.* **36**, 4519.

- [5] Izmet'sev A.A., Pogosyan G.S., Sissakian A.N., Winternitz P. (1996). *J. Phys. A: Math. Gen.* **29**, 5940.
- [6] Fialowski A., de Montigny M. (2005). *J. Phys. A: Math. Gen.* **38**, 6335.
- [7] Gilmore R. (1974). *Lie Groups, Lie Algebras and Some of Their Applications*, Wiley, New York.
- [8] Abe E. (1980). *Hopf Algebras, Cambridge Tracts in Mathematics 74*, Cambridge University Press, Cambridge.
- [9] Drinfeld V.G. (1986). *Quantum Groups in Proceedings of the International Congress of Mathematics*, MRSI, Berkeley, 798.
- [10] Tjin T. (1992). *Int. J. Mod. Phys. A* **7**, 6175.
- [11] Chari V., Pressley A. (1994). *A Guide to Quantum Groups*, Cambridge University Press, Cambridge.
- [12] Ballesteros A., Gromov N.A., Herranz F.J., del Olmo M.A., Santander M. (1995). *J. Math. Phys.* **36**, 5916.
- [13] Lukierski J., Nowicki A. (2003). *Int. J. Mod. Phys. A* **18**, 7.
- [14] Ahluwalia-Khalilova D.V. (2005). *Class. Quantum Grav.* **22**, 1433.
- [15] Herranz F.J. (2002). *Phys. Lett. B* **543**, 89.
- [16] Ballesteros A., Civitarese O., Herranz F.J., Reboiro M. (2002). *Phys. Rev. C* **66**, 064317.
- [17] Nijenhuis A., Richardson R.W. (1967). *J. Math. Mech.* **17**, 89.
- [18] Herranz F.J., Santander M. (2006). math-ph/0612059.
- [19] Ballesteros A., Herranz F.J., Ragnisco O. (2005). *Phys. Lett. B* **610**, 107.
- [20] Ragnisco O., Ballesteros A., Herranz F.J., and Musso F. (2007). *SIGMA* **3**, 026; math-ph/0611040.
- [21] Sommerville D.M.Y. (1910-11). *Proc. Edinburgh Math. Soc.* **28**, 25.
- [22] Yaglom I.M., Rozenfel'd B.A., Yasinskaya E.U. (1966). *Sov. Math. Surveys* **19**, 49.
- [23] Yaglom I.M. (1979). *A Simple Non-Euclidean Geometry and its Physical Basis*, Springer, New York.
- [24] Rozenfel'd B.A. (1988). *A history of non-euclidean geometry*, Springer, New York.
- [25] Man'ko V.I., Gromov N.A. (1992). *J. Math. Phys.* **33**, 1374.
- [26] Gromov N.A. (1992). *Contractions and Analytical Continuations of the Classical Groups. Unified Approach*, Komi Scienfic Center, Syktyvkar (in russian).
- [27] Santander M., Herranz F.J. (1997). *Int. J. Mod. Phys. A* **12**, 99.



- [28] Herranz F.J., Santander M. (1997). *J. Phys. A: Math. Gen.* **30**, 5411.
- [29] Ohn C. (1992). *Lett. Math. Phys.* **25**, 85.
- [30] de Montigny M., Patera J. (1991). *J. Phys. A: Math. Gen.* **24**, 525.
- [31] Moody R.V., Patera J. (1991). *J. Phys. A: Math. Gen.* **24**, 2227.
- [32] Herranz F.J., Santander M. (1996). *J. Phys. A: Math. Gen.* **29**, 6643.
- [33] Herranz F.J., de Montigny M., del Olmo M.A., Santander M. (1994). *J. Phys. A: Math. Gen.* **27**, 2515.
- [34] Wolf K.B., Boyer C.B. (1974). *J. Math. Phys.* **15**, 2096.
- [35] Bacry H., Lévy-Leblond J.M. (1968). *J. Math. Phys.* **9**, 1605.
- [36] Bacry H., Nuyts J. (1986). *J. Math. Phys.* **27**, 2455.
- [37] de Montigny M., Patera J., Tolar J. (1994). *J. Math. Phys.* **35**, 405.
- [38] Jordan C. (1961–1964). *Essai sur la géométrie à  $n$  dimensions*, Oeuvres, Gauthier–Villars, Paris.
- [39] Figueroa-O’Farrill J.M. (1989). *J. Math. Phys.* **30**, 2735.
- [40] Herranz F.J., Santander M. (2002). *J. Phys. A: Math. Gen.* **35**, 6601.
- [41] Izmet’shev A.A., Pogosyan G.S., Sissakian A.N., Winternitz P. (1999). *J. Math. Phys.* **40**, 1549.
- [42] Ballesteros A., Ragnisco O. (1998). *J. Phys. A: Math. Gen.* **31**, 3791.